Profunctor Optics, a Categorical Update (Extended Abstract)

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Abstract

Profunctor optics [PGW17, BG18] are a family of composable bidirectional data accessors. They provide a powerful abstraction over many data transformation patterns in functional programming described in libraries such as Kmett's *lens* library [Kme18]. Generalizing a result by Pastro and Street [PS08], we get a new proof of the equivalence between existential and profunctor representations of the optics. This extends to the case of *mixed optics* proposed by Riley [Ril18]. We collect derivations from the existential to the concrete form, including many original ones. In particular, we present an elementary derivation of the optic known as *traversal*, solving a problem posed by Milewski [Mil17]. We discuss a novel approach to composition of optics, based on both distributive laws and coproducts of monads. This is work in progress.

1 Optics

In functional programming, *optics* are a modular representation of bidirectional data accessors. Boisseau and Gibbons' profunctor representation theorem [BG18] proves that they can be equivalently written as functions polymorphic over profunctors. This profunctor representation is convenient because it turns composition of optics into ordinary function composition.

Example 1.1. *Lenses* are type-changing getter/setter pairs, defined as $\mathbf{Lens}((A,B),(S,T)) := (S \to A) \times (S \times B \to T)$ for any four types A, B, S, T. *Prisms* are data accessors enabling alternatives, defined as $\mathbf{Prism}((A,B),(S,T)) = (S \to A + T) \times (B \to T)$ for any four types A, B, S, T. Both are optics, in the sense of the following Definition 1.2, which means they can be written in profunctor form, thanks to Theorem 1.3, and composed using ordinary function composition. The following code uses a *prism* (`postal`) to parse a string into a *postal address*. The prism is then composed with a lens that accesses the *street* subfield inside the *postal address* (`street`). The composite optic can view and update the *street* field inside the string.

```
let address = "45 Banbury Rd, OX1 3QD, Oxford"
address^.postal
  -- {Street: "45 Banbury Rd", Code: "OX1 3QD", City: "Oxford"}
address^.postal.street
  -- "45 Banbury Rd"
address^.postal.street %~ "7 Banbury Rd"
  -- "7 Banbury Rd, OX1 3QD, Oxford"
```

A first unified definition of *optic* was proposed by Milewski [Mil17], who also suggested its monoidal constraints. This definition has been presented using submonoids of endofunctors by Boisseau and Gibbons [BG18] and using monoidal actions by Riley [Ril18]. We extend this definition to what Riley [Ril18] suggested to call *mixed optics*; we also extend his proof to show that mixed optics for a pair of monoidal actions are morphisms defining a category. The main proof technique is (co)end calculus as described, for instance, by Loregian [Lor15].

Definition 1.2. Let $\mathbf{M}$ be a monoidal category and let $\mathbf{C}$ and $\mathbf{D}$ be two arbitrary categories. Let $\underline{(\_)}\colon \mathbf{M} \to [\mathbf{C},\mathbf{C}]$ and $\underline{(\_)}\colon \mathbf{M} \to [\mathbf{D},\mathbf{D}]$ be two strong monoidal functors. An **optic** from $(S,T) \in \mathbf{C} \times \mathbf{D}$ with *focus* on $(A,B) \in \mathbf{C} \times \mathbf{D}$ is an element of the coend

$$\mathbf{Optic}\,((A,B),(S,T)) := \int^{M \in \mathbf{M}} \mathbf{C}(S, \underline{M}A) \times \mathbf{D}(\underline{M}B, T),$$

For this extended definition, we present an analogue of Boisseau and Gibbons' profunctor representation theorem [BG18]. Its proof is based on Pastro and Street's study of *doubles* for monoidal categories [PS08]; however, it generalizes tensor products to arbitrary monoidal actions over two different categories.

Theorem 1.3 (Profunctor representation theorem, after Boisseau and Gibbons [BG18])**.** *In the conditions of Definition 1.2,*

$$\int_{P \in た} \mathbf{Sets}(P(A,B), P(S,T)) \cong \mathbf{Optic}((A,B),(S,T)),$$

where た is the category of Tambara modules for the actions of $\mathbf{M}$.

Our definition of *Tambara module* generalizes Tambara's original one [Tam06] to arbitrary pairs of monoidal actions. As Pastro and Street [PS08] showed for the original case, they can be equivalently described by as coalgebras for a comonad $\Theta\colon \mathbf{Prof}(\mathbf{C},\mathbf{D}) \to \mathbf{Prof}(\mathbf{C},\mathbf{D})$ defined on profunctors by

$$\Theta P(A,B) = \int_{M \in \mathbf{M}} P(\underline{M}A, \underline{M}B).$$

As a corollary, the category **Optic** is shown to be the full subcategory on representable profunctors of the co-Kleisli category of Θ. This opens the possibility of exploring two different ways of composing optics of different *families*. The first is to consider distributive laws between Pastro-Street comonads; the second is to consider products of comonads. We show that both, under suitable considerations, produce again Pastro-Street comonads. This technique can be used to get some optics present in the literature such as the *affine traversal* [PGW17], but also to produce some original ones. Composition of optics of different kinds is common in programming practice; but a justification of its correctness was missing from the literature.

2 Examples of optics

An important justification of Definition 1.2 is that it captures the common examples of optics that occur in programming. Milewski [Mil17] showed that *lenses*, *prisms* and *grates* fit the definition relying only on elementary applications of the Yoneda lemma. Boisseau and Gibbons [BG18] and then Riley [Ril18] have shown the same for other common optics. We address the problem of finding an elementary derivation of the *traversal*, as proposed by Milewski [Mil17]. A **traversal** from (S,T) with focus on (A,B) is an element of $\mathbf{C}\,(S, \sum_n A^n \times (B^n \to T))$. We describe *traversals* as the optic for *power series functors*, also called polynomial functors in one variable [Koc09]. This is related to the more common description of traversals as optics for *traversable functors* by a result of Jaskelioff and O'Connor [JO15] that characterizes traversables as coalgebras for a certain parameterized comonad.

Proposition 2.1. *Given some functor* $C \in [\mathbb{N}, \mathbf{C}]$ *from the discrete category of the natural numbers, we can define a* power series *functor* $F \colon \mathbf{C} \to \mathbf{C}$ *given by* $F(A) = \sum_{n \in \mathbb{N}} C_n \times A^n$*. This induces a monoidal action that we call* Series: $[\mathbb{N}, \mathbf{C}] \to [\mathbf{C}, \mathbf{C}]$. Traversals *are optics for this action* Series: $[\mathbb{N}, \mathbf{C}] \to [\mathbf{C}, \mathbf{C}]$.

Proof. Unfolding the definitions, we want to prove that

$$\int^{C \in [\mathbb{N}, \mathbf{C}]} \mathbf{C}\left(S, \sum_{n \in \mathbb{N}} C_n \times A^n\right) \times \mathbf{C}\left(\sum_{n \in \mathbb{N}} C_n \times B^n, T\right) \cong \mathbf{C}\left(S, \sum_{n \in \mathbb{N}} A^n \times (B^n \to T)\right).$$

The fact that there exists an isomorphism between the two sets, natural in A, B, S and T, is a consequence of continuity of the hom-functor and the Yoneda lemma. □

We collect novel derivations for many other optics. The following are some of them, together with their generating monoidal actions.

Name	Monoidal action	Concrete form
Glass	Product and exponential	$((S \to A) \to B) \to S \to T$
Unsorted Traversal	Combinatorial species	$S \to \int^{n \in \mathbf{B}} A^n \times (B^n \to T)$
Algebraic lens	Product by a ψ-algebra	$(S \to A) \times (\psi S \times B \to T)$
Kaleidoscope	Applicative functors	$\prod_{n \in \mathbb{N}} (A^n \to B) \to (S^n \to T)$

The generalization to *mixed optics* allows us to consider *degenerate optics*. These are optics where one of the categories is the terminal category. Degenerate optics include *getters*, *setters* and *folds*; as they appear in Kmett's *lens* library [Kme18]. This definition also captures some variants of lenses and, remarkably, a generalization of lenses to an arbitrary monoidal category proposed by Myers and Spivak [Spi19, §2.2].

3 A case study

Let us discuss an example of how our results can be used in practice. Consider the `iris` dataset [Fis36], where each entry represents a *flower* described by its species and four real number measurements.

Example 3.1. An *algebraic lens* (`measurements`) for the list monad is used first as an ordinary lens to access the first element of the dataset (line 1), and then to encapsulate some learning algorithm that classifies measurements into a species (line 2). Consider a *kaleidoscope* that extends an aggregating function on the reals to the measurements (`aggregateWith`). Our work has shown that both fit Definition 1.2, which allows us to use Theorem 1.3 and our results on composition of optics to join them into a new kaleidoscope (`measurements.aggregateWith`, in line 3). It is remarkable that we just use ordinary function composition.

```
(iris !! 1)^.measurements
-- (4.9, 3.0, 1.4, 0.2)
iris ?. measurements (Measurements 4.8 3.1 1.5 0.1)
-- Iris Setosa (4.8, 3.1, 1.5, 0.1)
iris >- measurements.aggregateWith mean
-- Iris Versicolor (5.8, 3.0, 3.7, 1.1)
```

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